BACHELOR THESIS TOPICS IN TOPOLOGY

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1. Vector fields on spheres

The vector fields problem, solved by Adams in [Ada62], is quite easy to state. Given a positive integer n, determine the maximal number $k(n)$ of linearly independent (tangent) vector fields on the unit sphere S^{n-1} in $\mathbb{R}^n$. Here a vector field is just a continuous function $V : S^{n-1} \to \mathbb{R}^n$ such that $x \perp V(x)$ for all $x \in S^{n-1}$ and linear independence is defined pointwise.

It had already been known for quite some time how many vector fields could be constructed algebraically (i.e. as the restriction of linear transformations $x \mapsto Ax$). By the Hurwitz-Radon-Eckmann theorem ([Eck43]) there are $\rho(n) - 1$ linear vector fields (and no more), where the *Radon-Hurwitz numbers* $\rho(n)$ are defined as follows: Writing $n = u \cdot 2^{4\alpha+\beta}$ with u odd and $0 \leq \beta < 4$, we set $\rho(n) = 8\alpha + 2^\beta$. This can also be phrased in terms of *Clifford algebras* ([ABS64], [LM89]). A set of k linear vector fields amounts to an n-dimensional representation of the $\mathbb{R}$-algebra Cl_k with generators $e_1, \ldots, e_k$ subject to the relations $e_i e_j + e_j e_i = -2 \cdot \delta_{i,j}$. The representation theory of Clifford algebras is well understood and from this one reads off the Radon-Hurwitz numbers combinatorially. The first goal of this project is to explain this algebraic part of the story.

Now it is a much deeper fact that $\rho(n) - 1$ is also an upper bound for the number of all vector fields ([Ada62]) and this is where algebraic topology enters the picture. A set of k vector fields on S^{n-1} can equivalently be described as a section of the projection $V_{k+1,n} \to S^{n-1}$ from the *Stiefel manifold* of $(k+1)$-frames in $\mathbb{R}^n$ to the sphere. For the purposes of this problem the Stiefel manifold can be approximated by the stunted projective space $\mathbb{R}P^{n-1}/\mathbb{R}P^k$ and it should be discussed how the topology of these spaces yields some upper bounds, for example using Steenrod operations ([MT68]). Adams solution was ultimately based on calculating the KO-theory of stunted projective spaces, where KO is a cohomology theory defined in terms of real vector bundles, called real (or orthogonal) K-theory (see [Ati67] for a textbook introduction to the complex version). A more ambitious version of this bachelor thesis project would be to sketch a streamlined approach to his result involving the so-called J-homomorphism $KO(\mathbb{R}P^n) \to J(\mathbb{R}P^n)$ ([Jam76],[Bot69],[Bot62], [ABS64], also see [Ati67] for the complex case). This would then be the main focus and one should only spend very little time on the classical algebraic story.

2. Computation of the unoriented cobordism ring

The (unoriented) cobordism ring Ω_* is the graded ring of cobordism classes of smooth, closed manifolds with addition and multiplication induced by disjoint union and cartesian product. Two manifolds (of the same dimension) are said to be *cobordant* if their disjoint union is the boundary of a manifold and the grading is given by dimension. As miraculous as it may seem, this can actually be computed. The *Pontryagin-Thom construction* (e.g. see [BtD70] or [Mil65] for the 'framed' version) allows one to translate this into a problem in (stable) homotopy theory. Given a submanifold $M^n \subset \mathbb{R}^{n+k}$, one can embed its normal bundle $\nu \subset \mathbb{R}^{n+k}$ as an open neighbourhood. Sending the complement of ν to the point at infinity yields the Pontryagin-Thom collapse map $S^{n+k} \to \mathrm{Th}(\nu)$ to the *Thom space* $\mathrm{Th}(\nu)$ which is simply the one-point compactification of ν. The (k-dimensional) normal bundle is classified by a map $M \to BO(k)$ and this gives a map $\mathrm{Th}(\nu) \to \mathrm{Th}(\gamma_k) = MO_k$ to the Thom space of the universal vector bundle γ_k over $BO(k)$. These Thom spaces come with suspension maps $MO_k \wedge S^1 \to MO_{k+1}$ turning the entire collection into a *spectrum* MO and it turns out that the Pontryagin-Thom construction determines a well-defined morphism $\Omega_n \to \mathrm{colim}_k\, \pi_{n+k} MO_k = \pi_n MO$ to the stable homotopy groups of MO. In fact this is an isomorphism and Thom famously also computed the result ([Tho54])

$$\Omega_* \cong \pi_* MO \cong \mathbb{F}_2[x_i \mid i \neq 2^k - 1]$$

to be a polynomial ring over $\mathbb{F}_2$ on countably many generators x_i of certain degrees i (there are also explicit manifold representatives for these, see [Dol56]).

The main goal of this project is to work out this computation, for example following this very rough outline: Using the *Thom-isomorphism*, it is relatively straightforward to compute the mod 2 (co)homology of these Thom spaces and hence of MO. One ends up with a polynomial ring $H_*MO \cong \mathbb{F}_2[b_1, b_2, \ldots]$ on generators of all degrees. The next step is to determine the structure as a comodule over the *dual Steenrod algebra* ([Mil58], [MT68]) $\mathcal{A}_*$ and it turns out that it is coinduced: $H_*MO \cong \mathcal{A}_* \otimes \mathbb{F}_2[\tilde{x}_i \mid i \neq 2^k - 1]$. This is a truly stable phenomenon and cannot happen for the homology of a single space. In any case, by dualizing we see that the cohomology is free as a module over the Steenrod algebra. This together with the fact $\pi_0 MO \cong \mathbb{F}_2$ allows one to conclude that the stable mod 2 Hurewicz map $\pi_*MO \to H_*MO$ induces an isomorphism with the suggestively denoted tensor factor. There are many references for the computation of the cobordism ring (e.g. [Sto68], [BtD70], [Swi75] as well as various lecture notes). This topic is also nicely extendable. For example it would also be nice to say something about manifold generators. Moving into more optional territory, there is a fascinating description of these results in terms of formal group laws (in particular expressing the comodule structure of the homology in a more conceptual way).

References

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