Constructions of Derived Equivalences of Finite Posets

Sefi Ladkani

Einstein Institute of Mathematics
The Hebrew University of Jerusalem

http://www.ma.huji.ac.il/~sefil/

Notions

X – *Poset* (finite partially ordered set).

The *Hasse diagram* G_X of X is a directed acyclic graph.

- Vertices: the elements $x \in X$.
- Edges $x \to y$ for pairs $x < y$ with no z such that $x < z < y$.

X carries a natural *topology*:

$U \subseteq X$ is open if $x \in U\,,\, y \geq x \Rightarrow y \in U$

We get a finite T_0 topological space.

Equivalence of notions:

Posets $\Leftrightarrow$ Finite T_0 spaces

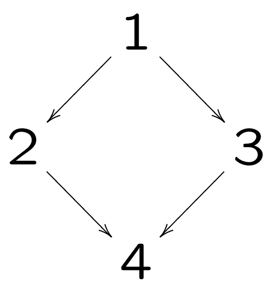
For a field k, the *incidence algebra* kX of X is a matrix subalgebra spanned by e_{xy} for $x \leq y$.

Example

Poset $X = \{1, 2, 3, 4\}$ with

$$1 < 2, \;\; 1 < 3, \;\; 1 < 4, \;\; 2 < 3, \;\; 2 < 4, \;\; 3 < 4$$

Hasse diagram



Topology
The open sets are:

$$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$$

Incidence algebra ($*$ can take any value)

$$\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

Three Equivalent Categories

$\mathcal{A}$ – Abelian category.

• *Sheaves* over X with values in $\mathcal{A}$:

$$U \mapsto \mathcal{F}(U) \qquad\qquad U \subseteq X \text{ open}$$

with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ $(U \supseteq V)$, pre-sheaf and gluing conditions.

• *Commutative diagrams* of shape G_X over $\mathcal{A}$, or *functors* $X \to \mathcal{A}$:

$$F_x \xrightarrow{r_{xy}} F_y \qquad\qquad x \to y$$

with $r_{xy} \in \text{hom}_{\mathcal{A}}(F_x, F_y)$ and commutativity relations.

Fix a field k, and specialize:
$\mathcal{A}$ – finite dimensional vector spaces over k

• Finitely generated *right modules* over the *incidence algebra* of X over k.

The Problem

$\mathcal{D}^b(X)$ – *Bounded derived category* of sheaves / diagrams / modules (over X).

Two posets X, Y are *equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$$

Problem. When $X \sim Y$ for two posets X, Y?

No known algorithm that decides if $X \sim Y$; however one can use:

• *Invariants* of the derived category;
If $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ then X and Y must have the same invariants.

Examples of invariants are:

- The *number of points* of X.
- The *Euler bilinear form* on X.

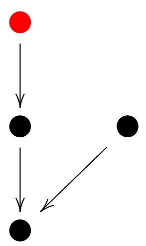
• *Constructions*
Start with some "nice" X and get many Y-s with $X \sim Y$.

Known Constructions

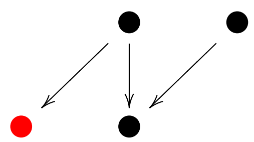
- **BGP Reflection**

When X is a tree and $s \in X$ is a *source* (or a *sink*), invert all arrows from (to) s and get a new tree X' with $X' \sim X$.

Example.

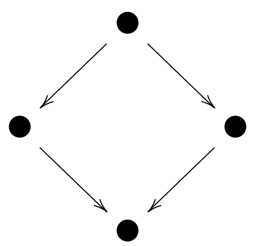


and

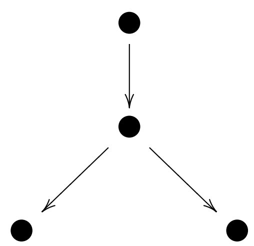


are equivalent.

- **The square and D_4**



and



are equivalent.

New Construction

A few definitions

Given a poset S, denote by S^{op} the *opposite poset*, with $S^{op} = S$ and $s \leq s'$ in S^{op} if and only if $s \geq s'$ in S.

A poset S is called a *bipartite graph* if we can partition $S = S_0 \amalg S_1$ with S_0, S_1 discrete with the property that $s < s'$ in S implies $s \in S_0$, $s' \in S_1$.

Let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by the elements of another poset S.

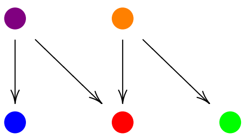
The *lexicographic sum of the X_s along S*, denoted $\oplus_S \mathfrak{X}$, is a new poset $(X, \leq)$;
Its elements are $X = \coprod_{s \in S} X_s$, with the order $x \leq y$ for $x \in X_s$, $y \in X_t$ if either $s < t$ (in S) or $s = t$ and $x \leq y$ (in X_s).

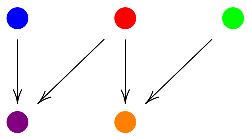
New Construction – Theorem

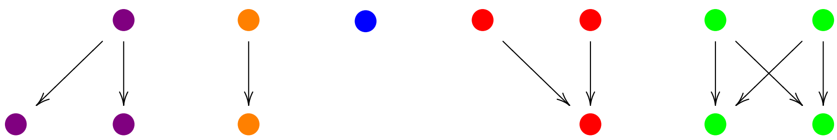
Theorem.
If S is a bipartite graph and $\mathfrak{X} = \{X_s\}_{s \in S}$ is a collection of posets, then

$$\oplus_S \mathfrak{X} \sim \oplus_{S^{op}} \mathfrak{X}$$

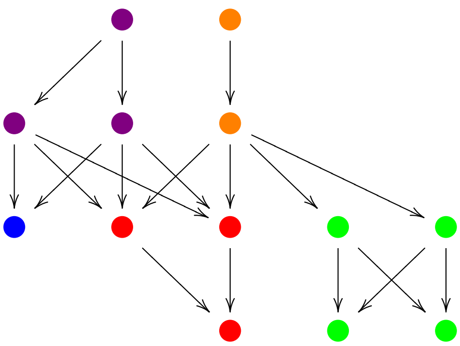
Example.

$S =$ 

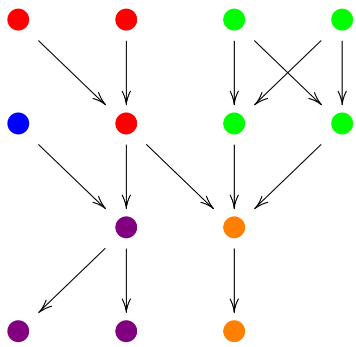
 $= S^{op}$

$\mathfrak{X} =$ 

$\oplus_S \mathfrak{X}$



$\oplus_{S^{op}} \mathfrak{X}$

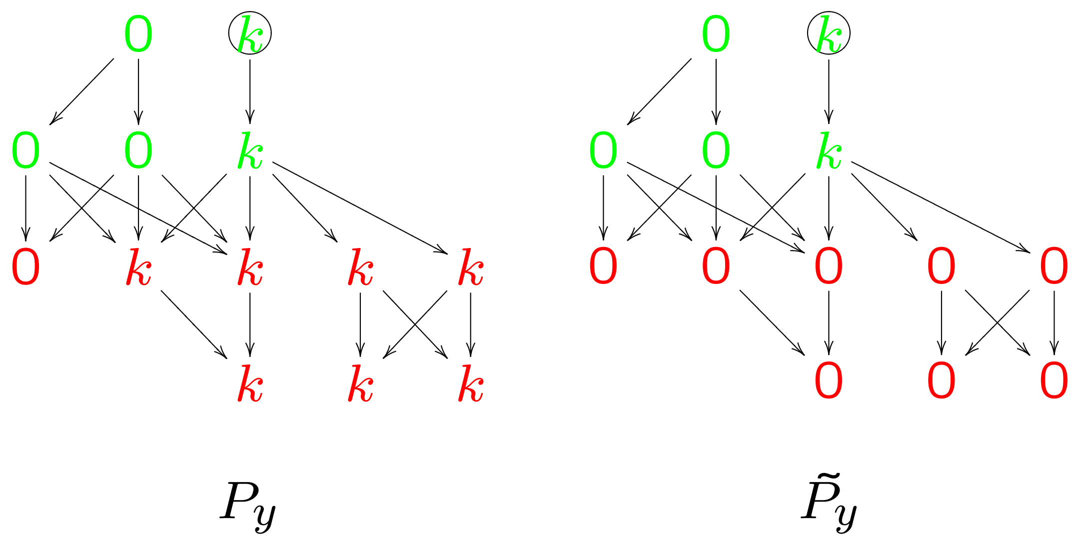


Idea of the Proof

Let $Y \subset X$ be closed, $U = X \setminus Y$. Denote by $i : Y \to X$, $j : U \to X$ the inclusions.

Consider the truncations $\tilde{P}_y = i_* i^{-1} P_y$, $\tilde{I}_u = j_! j^{-1} I_u$ for $y \in Y$, $u \in U$.

Example. $X = Y \cup U$.



Then $\{\tilde{P}_y\}_{y \in Y} \cup \{\tilde{I}_u[1]\}_{u \in U}$ is a *strongly exceptional collection* in $\mathcal{D}^b(X)$, hence

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(A_Y)$$

where $A_Y = \mathrm{End}_{\mathcal{D}^b(X)}((\oplus_Y \tilde{P}_y) \oplus (\oplus_U \tilde{I}_u)[1])$.

Proof – continued

k-basis of the algebra A_Y

$$\left\{e_{yy'} : y \leq y'\right\} \cup \left\{e_{u'u} : u' \leq u\right\} \cup \{e_{uy} : y < u\}$$

where $y, y' \in Y$, $u', u \in U$.

Multiplication formulas

$e_{yy'}e_{y'y''} = e_{yy''}$, $e_{u''u'}e_{u'u} = e_{u''u}$

$e_{uy}e_{yy'} = e_{uy'}$ if $y' < u$ and 0 otherwise.
$e_{u'u}e_{uy} = e_{u'y}$ if $y < u'$ and 0 otherwise.

Define a *binary relation* $\leq'$ on $X' = U \amalg Y$ by

$$u' \leq' u \Leftrightarrow u' \leq u \qquad y \leq' y' \Leftrightarrow y \leq y'$$
$$u <' y \Leftrightarrow y < u$$

$\leq'$ is a *partial order* if and only if

$$y \leq y' \in Y, u' \leq u \in U, y < u \Rightarrow y' < u'$$

In this case, the algebra A_Y is isomorphic to the *incidence algebra* of $(X', \leq')$.

Ordinal Sums

Corollary. $X \oplus Y \sim Y \oplus X$.

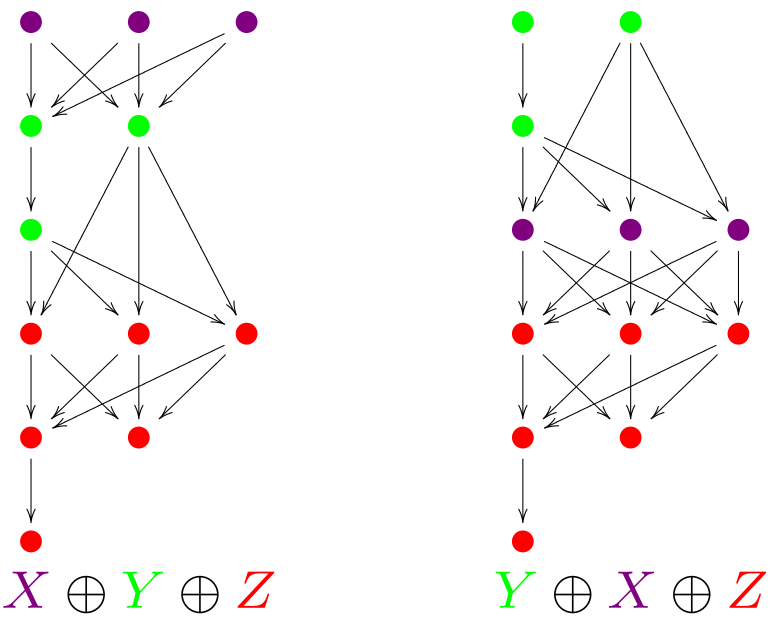
Proposition. Assume that for any X, Y, Z,

$$(\star) \qquad X \oplus Y \oplus Z \sim Y \oplus X \oplus Z$$

Then, for all $X_1, \ldots, X_n$ and $\pi \in S_n$,

$$X_{\pi(1)} \oplus \cdots \oplus X_{\pi(n)} \sim X_1 \oplus \cdots \oplus X_n$$

Counterexample to $(\star)$.



are *not* equivalent!