Derived Equivalences of Categories of Sheaves over Finite Partially Ordered Sets

Sefi Ladkani

Einstein Institute of Mathematics
The Hebrew University of Jerusalem

Introduction

The purpose of my research is to explore the bounded derived categories $\mathcal{D}^b(X)$ of diagram categories over finite posets X.

Applications and Related areas:

1. **(Geometry)** Computation of the cohomology of subspace arrangements [3].

2. **(Combinatorics)** Study of h-vectors of convex polytopes [4].

3. **(String theory)** Homological mirror symmetry [5].

4. **(Algebraic geometry)** Study of derived categories of coherent sheaves over algebraic varieties [2];
   Non-commutative geometry.

Posets

A *poset* $(X, \leq)$ is a set X with a binary relation $\leq$ satisfying

(reflexive)	$x \leq x$
(anti-symmetric)	$x \leq y, y \leq x \Rightarrow x = y$
(transitive)	$x \leq y, y \leq z \Rightarrow x \leq z$

Examples:

1. The set of natural numbers with the usual order: $0 < 1 < 2 < 3 < \dots$.

2. The set of integers with the division relation: $a \leq b$ if a divides b.

3. The set $\mathcal{P}(Y)$ of all subsets of a given set Y with the inclusion relation: $S \leq T$ if $S \subseteq T$.

$$\phi \leq \{a\} \leq \{a, b\} \quad , \quad \phi \leq \{b\} \leq \{a, b\}$$

Hasse Diagrams

Given a finite poset $(X, \leq)$, its *Hasse diagram* is a directed graph;

- Its vertices are the elements $x \in X$.
- Its edges $x \to y$ are pairs $x < y$ such that no z satisfies $x < z < y$.

Examples:

1. The natural numbers:

$$0 \to 1 \to 2 \to 3 \to \ldots$$

2. $\mathcal{P}(\{a, b, c\})$:

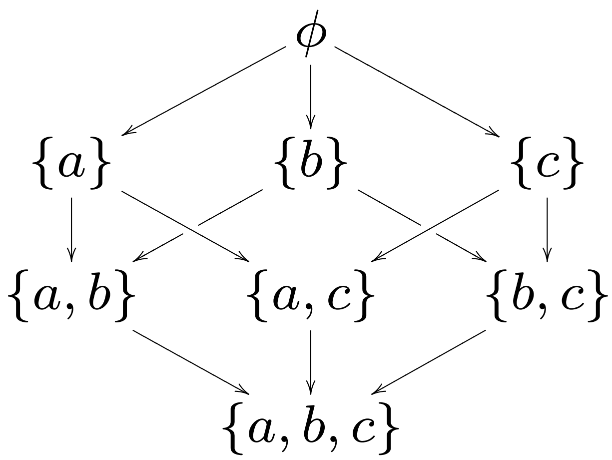


Diagram Categories

Let $(X, \leq)$ be a finite poset (as a Hasse diagram) and let k be a field.

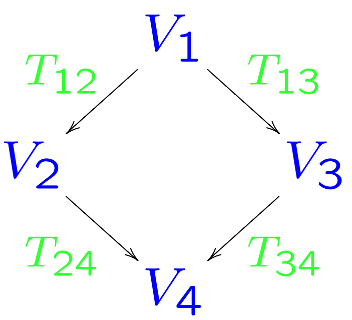
The *diagram category* over X consists of objects and morphisms.

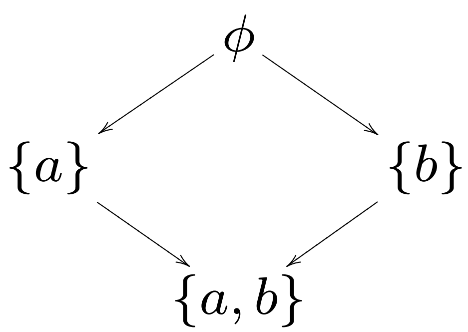
An *object* consists of:

- Finite dimensional vector space V_x for each vertex $x \in X$.
- Linear transformation $T_{xy} : V_x \to V_y$ for each edge $x \to y$.

We require that the composition of the linear transformations along a path depends only on its starting and ending points.

Example. $\mathcal{P}(\{a, b\})$. An object is a diagram below with $T_{24}T_{12} = T_{34}T_{13}$.





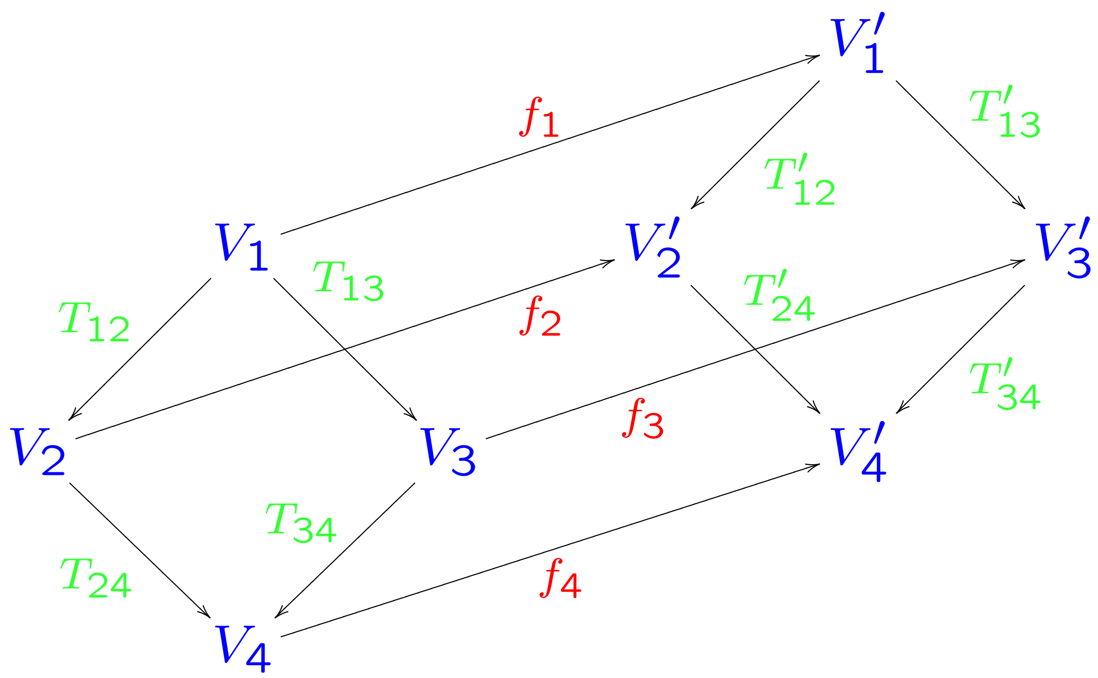
A *morphism* between two objects $\{V_x, T_{xy}\}$, $\{V'_x, T'_{xy}\}$ consists of linear transformations

$$f_x : V_x \to V'_x$$

for each vertex $x \in X$, such that for any edge $x \to y$,

$$f_y T_{xy} = T'_{xy} f_x$$

Example. $\mathcal{P}(\{a, b\})$. A morphism is a tuple (f_1, f_2, f_3, f_4) such that all squares in the following diagram are commutative.



Topology and Algebra

Define a *topology* on X by:

$U \subseteq X$ is open if $x \in U$, $y \geq x \Rightarrow y \in U$

The *incidence algebra* A_X of X is a matrix subalgebra generated by E_{xy} for $x \leq y$.

Example. $\mathcal{P}(\{a, b\})$. The incidence algebra is: ($*$ can take any value)

$$\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

The open sets are:

$$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$$

Three equivalent notions:

Diagrams on X (finite poset)

Sheaves on X (topology as above)

(Right) finite dimensional *modules* over A_X

The Derived Category

A *complex* of diagrams is a sequence of diagrams $\mathcal{F}_n$ and morphisms $d_n : \mathcal{F}_n \to \mathcal{F}_{n+1}$

$$\dots \to \mathcal{F}_{-1} \xrightarrow{d_{-1}} \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \to \dots$$

such that $d_{n+1} d_n = 0$ for all n.

A complex is *bounded* if $\mathcal{F}_n = 0$ for all but finite number of n.

Complexes also form a category.

The *derived category* is obtained by taking complexes modulo a suitable equivalence relation (*quasi-isomorphism*).

We will focus on the *bounded* derived category corresponding to bounded complexes of diagrams on X, and denote it by $\mathcal{D}^b(X)$.

The Problem

Two posets X, Y are *equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$$

Problem. When $X \sim Y$ for two posets X, Y?

No known algorithm that decides if $X \sim Y$; however one can use:

Invariants of the derived category;
If $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ then X and Y must have the same invariants.

Examples of invariants are:

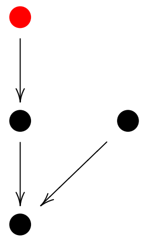
- The *number of points* of X.
- The *Euler bilinear form* on X.

Constructions
Start with some "nice" X and get many Y-s with $X \sim Y$.

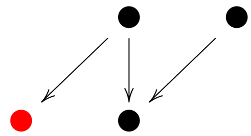
Known Constructions

BGP Reflection [1]
When X is a tree and $s \in X$ is a *source* (or a *sink*), invert all arrows from (to) s and get a new tree X' with $X' \sim X$.

Example.

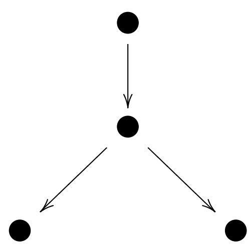


and



are equivalent.

D_4 **and the square**



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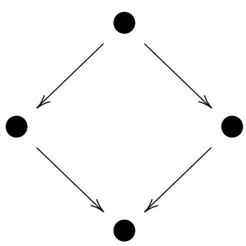


are equivalent.

New Construction

A few definitions

Given a poset S, denote by S^{op} the *opposite poset*, with $S^{op} = S$ and $s \leq s'$ in S^{op} if and only if $s \geq s'$ in S.

A poset S is called a *bipartite graph* if we can partition $S = S_0 \coprod S_1$ with S_0, S_1 discrete with the property that $s < s'$ in S implies $s \in S_0$, $s' \in S_1$.

Let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by the elements of another poset S.

The *lexicographic sum of the X_s along S*, denoted $\oplus_S \mathfrak{X}$, is a new poset $(X, \leq)$;
Its elements are $X = \coprod_{s \in S} X_s$, with the order $x \leq y$ for $x \in X_s$, $y \in X_t$ if either $s < t$ (in S) or $s = t$ and $x \leq y$ (in X_s).

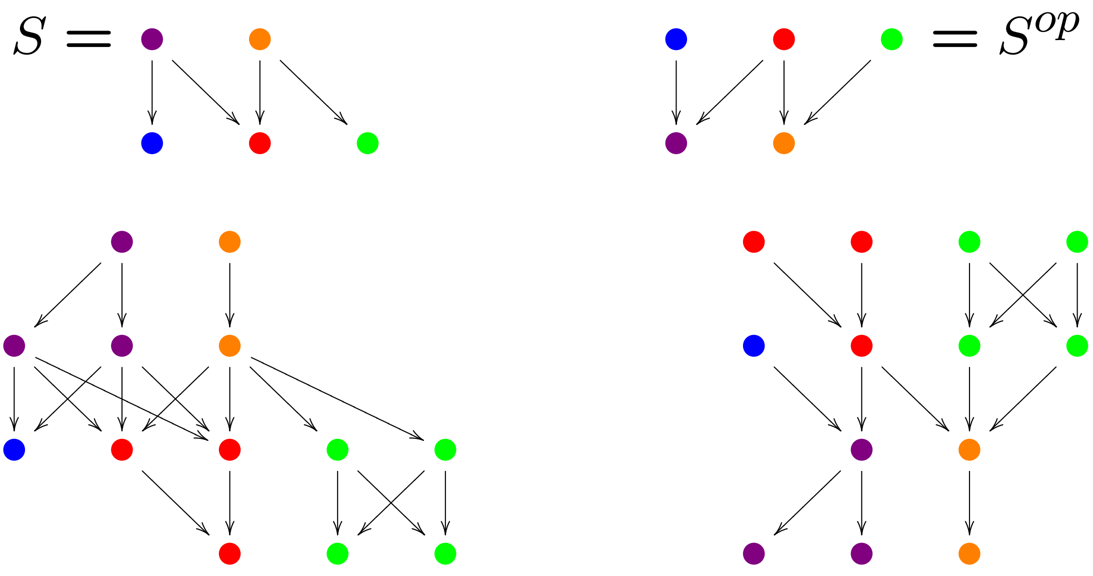
New Construction – Theorem

Theorem.
If S is a bipartite graph and $\mathfrak{X} = \{X_s\}_{s \in S}$ is a collection of posets, then

$$\oplus_S \mathfrak{X} \sim \oplus_{S^{op}} \mathfrak{X}$$

This theorem generalizes some of the known constructions.

Example.



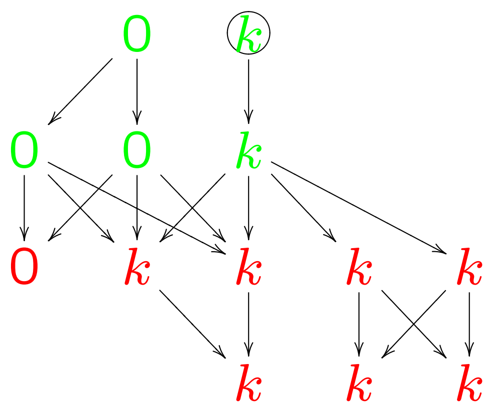
Corollary. $X \oplus Y \sim Y \oplus X$

Idea of the Proof

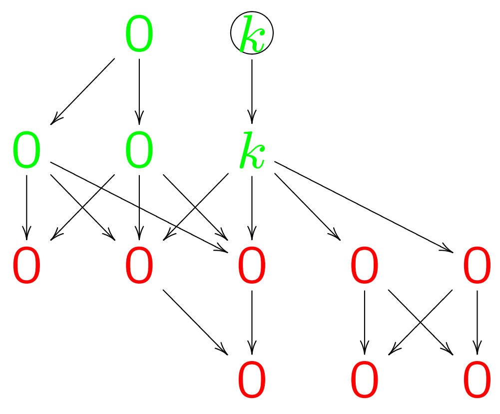
Let $Y \subset X$ be closed, $U = X \setminus Y$. Denote by $i : Y \to X$, $j : U \to X$ the inclusions.

Consider the truncations $\tilde{P}_y = i_* i^{-1} P_y$, $\tilde{I}_u = j_! j^{-1} I_u$ for $y \in Y$, $u \in U$.

Example. $X = Y \cup U$.



P_y



$\tilde{P}_y$

Then $\{\tilde{P}_y\}_{y \in Y} \cup \{\tilde{I}_u[1]\}_{u \in U}$ is a *strongly exceptional collection* in $\mathcal{D}^b(X)$, hence

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(A)$$

where $A = \mathrm{End}_{\mathcal{D}^b(X)}((\oplus_Y \tilde{P}_y) \oplus (\oplus_U \tilde{I}_u)[1])$.

Choose Y such that A is an incidence algebra, and then identify its underlying poset.

A Generalization?

Question. Is the theorem also true for posets S with 3 layers?

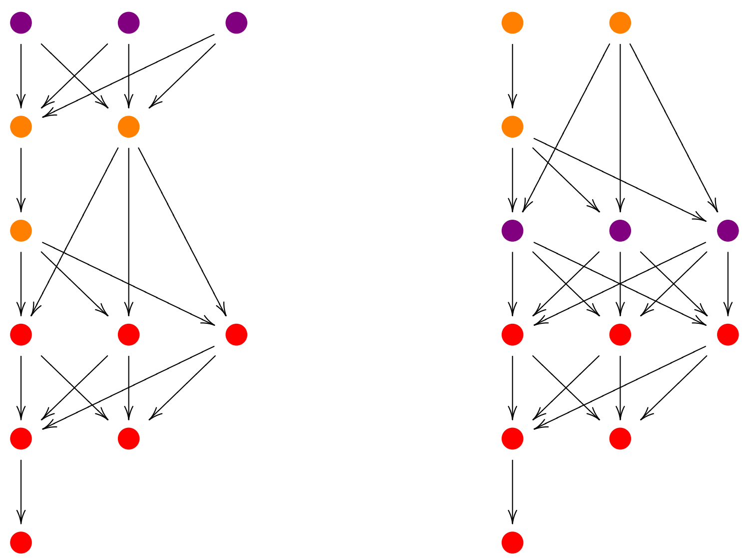
The simplest case to consider is the ordinal sum of three posets: $X \oplus Y \oplus Z$.

Note that

$$X \oplus Y \oplus Z \sim Y \oplus Z \oplus X \sim Z \oplus X \oplus Y$$
$$Y \oplus X \oplus Z \sim X \oplus Z \oplus Y \sim Z \oplus Y \oplus X$$

(why?)

Counterexample.



$X \oplus Y \oplus Z$ $\qquad$ $Y \oplus X \oplus Z$

are *not* equivalent!

References

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[2] Bondal A., Orlov D. *Derived Categories of coherent sheaves.* Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 47–56.

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