Exercises in Geometry II

University of Bonn, Summer Semester 2018
Dozent: PD Dr. Fernando Galaz-Garcia
Assistant: Saskia Roos
Sheet 4

Rheinische Friedrich-Wilhelms-Universität Bonn

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1. Jacobi fields along geodesics [4 points]

Let (M, g) be a Riemannian manifold with constant sectional curvature C and let γ be a unit speed geodesic in M.
Show that the normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields

$$J(t) = u(t)E(t),$$

where E is any parallel normal vector field along γ, and $u(t)$ is given by

$$u(t) = \begin{cases} t, & \text{if } C = 0, \\ R\sin\left(\frac{t}{R}\right), & \text{if } C = \frac{1}{R^2} > 0, \\ R\sinh\left(\frac{t}{R}\right), & \text{if } C = -\frac{1}{R^2} < 0. \end{cases}$$

2. Taylor series of Riemannian metrics [4 points]

Let (M, g) be a Riemannian manifold and fix a point $p \in M$. Show that the second order Taylor series of g is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{n} R_{iklj}x^k x^l + O(|x|^3),$$

in Riemannian normal coordinates $(x_1, \ldots, x_n)$ centered at p.
Hint: Consider a radial geodesic $\gamma(t) = (tv_1, \ldots, tv_n)$ and a Jacobi field $J(t) = tW^i\partial_i$ along γ. Compute the first four t-derivatives of $|J(t)|^2$ at $t = 0$ in two different ways using the Jacobi equation.

3. Conjugate points [4 points]

Let (M, g) be a complete Riemannian manifold and let $SM := \{(x, v) \in TM : \|v\| = 1\}$ denote the unit tangent bundle. Given $(x, v) \in SM$, we let γ_v be the the geodesic with $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. For all $(x, v) \in SM$ we define $\mathrm{con}(v) \in (0, \infty]$ to be the first $t > 0$ such that $\gamma_v(t)$ is a conjugate point to $\gamma(0)$. Show that $\mathrm{con}(-\dot\gamma_v(\mathrm{con}(v))) = \mathrm{con}(v)$ holds for all $(x, v) \in SM$.

4. Jacobi fields on manifolds with non-positive sectional curvature [4 points]

Let (M, g) be a Riemannian manifold with non-positive sectional curvature.

a) Let J be a Jacobi field along a differentiable curve $\gamma\colon [a, b] \to M$. Show that $f(t) := \|J(t)\|^2$ is a convex function, i.e. $f''(t) \geq 0$ for all t.

b) Conclude from a) that M has no conjugate points.

Due on Monday, May 21.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`