Location of the Spectrum of Operator Matrices which are Associated to Second-Order Systems

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Second-order systems: Framework

$$\begin{aligned} \ddot{z}(t) + A_0 z(t) + D\dot{z}(t) &= B_0 u(t), \quad t \geq 0, \\ z(0) &= z_0, \quad \dot{z}(0) = w_0, \\ y(t) &= B_0^* z(t) \end{aligned}$$

- $A_0 : D(A_0) \subset H \to H$ is selfadjoint with $\langle A_0 x, x \rangle \geq \gamma \|x\|^2$.

  $H_{1/2} = D(A_0^{1/2})$ with $\|x\|_{1/2} := \|A_0^{1/2} x\|$, $\quad H_{-1/2} = \overline{H}^{\|A_0^{-1/2} \cdot \|}$

  A_0 can be extended to an operator: $H_{1/2} \to H_{-1/2}$
- $D \in \mathcal{L}(H_{1/2}, H_{-1/2})$ with $A_0^{-1/2} D A_0^{-1/2} \in \mathcal{L}(H)$ selfadjoint and non-negative.
- $B_0 \in \mathcal{L}(\mathbb{C}^m, H_{-1/2})$.
- $\exists \beta > 0 : \langle Dz, z \rangle_{H_{-1/2} \times H_{1/2}} \geq \beta \|B_0^* z\|^2, \quad \forall z \in H_{1/2}$.

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is equivalent to

$$\dot{x}(t) = Ax(t) + Bu(t)$$
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$$A = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_0 \end{pmatrix}, \quad C = \begin{pmatrix} B_0^* & 0 \end{pmatrix}$$

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$$D(A) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in H_{1/2} \times H_{1/2} \mid A_0 z + Dw \in H \right\}$$

Th.[BI'88, L'89, CLL'98, HS'03, TW'03]:
A generates a C_0-semigroup of contractions on $H_{1/2} \times H$.

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Topics of this Talk

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- spectrum of A,
- essential spectrum,
- intervals with no accumulation of non-real spectrum,
- no spectrum on $i\mathbb{R}$
- Example and Minimum Phase

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Spectrum of the operator A

First result
Th.[TW'03]:

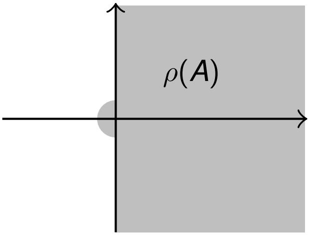
$$\{0\} \cup \mathbb{C}_0 \subset \rho(A),$$

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

hence

$$\sigma(A) = \overline{\sigma(A)}.$$



Question: Spectrum of A arbitrary in the closed left half plane?

Spectrum of the operator A

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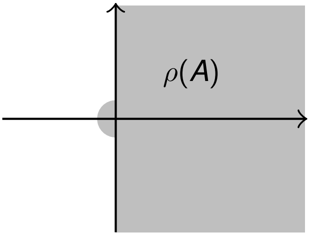


hence

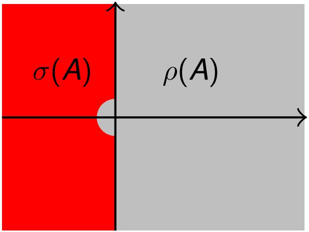
$$\sigma(A) = \overline{\sigma(A)}.$$

Question: Spectrum of A arbitrary in the closed left half plane?

Spectrum of A arbitrary?

Answer: YES.

Th.:
For each $\varepsilon > 0$ there exist an uniformly positive operator A_o and a operator D such that $\sigma(A) = \{s \in \overline{\mathbb{C}_-} \mid |s| \geq \varepsilon\}$.



Therefore: We have to impose additionally conditions.

In detail:

Let $H = L^2(0,\infty)$ and let $(q_j) \subset \mathbb{R}$ with $\overline{(q_j)} = \mathbb{Q}$. Set

$$a(x) := q_j \quad \text{if} \quad j-1 \leq x < j, \quad j \in \mathbb{N},$$

$$d(x) := \begin{cases} \frac{1}{x-j+1} - 1 & \text{if } j-1 \leq x < j \text{ and } |q_j| \geq \varepsilon, \\ \frac{1}{x-j+1} - 1 + \sqrt{\varepsilon^2 - q_j^2} & \text{if } j-1 \leq x < j \text{ and } |q_j| < \varepsilon. \end{cases}$$

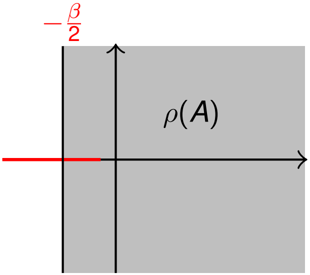
Define $a_o(x) := a(x)^2 + d(x)^2$, $x \in [0,\infty)$.

$$\begin{aligned} (A_o f)(x) &:= a_o(x) f(x), \quad x \in [0,\infty), \\ (Df)(x) &:= 2d(x) f(x), \quad x \in [0,\infty), f \in H_{1/2}, \end{aligned}$$

Spectrum of the operator A

Th.[BI'88, CLL'98, HS'03, TW'03]:
Assume $\langle Dz, z\rangle \geq \beta ||z||^2$, "$D$ is large." Then

$$\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > -\frac{\beta}{2}, \mathrm{Im}\,\lambda \neq 0\} \subset \rho(A).$$

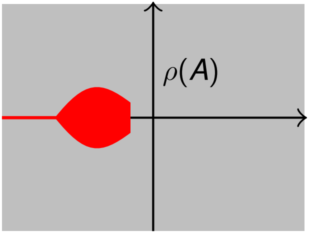


Spectrum of the operator A

Th.:
Assume $\langle Dz, z\rangle \geq \gamma ||z||^2_{H_{1/2}} \geq \gamma ||z||^2$, "$D$ is very large."
Then

Then there exists $a < 0$:

$$\sigma(A) \subset (-\infty, a] \cup \{\lambda \mid \mathrm{Re}\,\lambda \leq a,\ (\mathrm{Im}\,\lambda)^2 \leq -2\gamma \mathrm{Re}\,\lambda - (\mathrm{Re}\,\lambda)^2\}$$



Spectrum and essential spectrum of the operator A

Th.:
Set

$$\alpha_1 := \frac{1}{2\|A_o^{-1}\|} \min \sigma_{ess}(A_o^{-1}D).$$

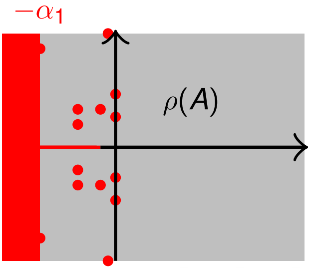
Then

$$\sigma_{ess}(A) \subset (-\infty, 0) \cup \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \leq -\alpha_1\}.$$

Non-real spectrum does not accumulate to

$$(-\alpha_1, 0)$$

.



Spectrum and essential spectrum of the operator A

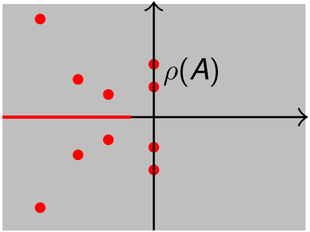
Th. :
Assume A_0^{-1} is compact Then

$$\sigma_{\text{ess}}(A) = \{\lambda \mid \lambda^{-1} \in \sigma_{\text{ess}}(-A_0^{-1}D)\}$$

$$\sigma(A) = \sigma_{\text{ess}}(A) \cup \sigma_{p,\text{norm}}(A)$$

$\sigma_{\text{ess}}(A)$ is not an accumulation point of

$$\sigma(A) \cap (\mathbb{C} \backslash \mathbb{R})$$



Spectrum and essential spectrum of the operator A

Th. [CLL'98]:
Assume A_0^{-1} is compact and
$\langle Dz, z\rangle > 0$ for any eigenvector z of A_0. Then

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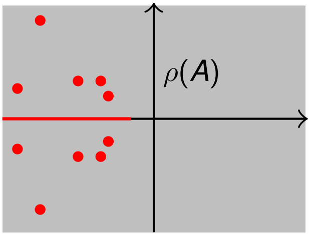
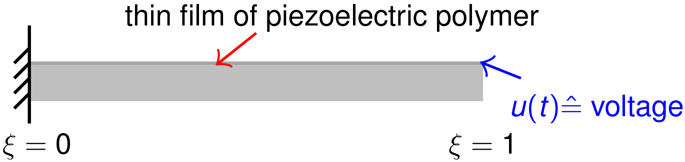
$$\sigma(A) \cap i\mathbb{R} = \emptyset$$



Euler-Bernoulli beam equation: Kelvin-Voigt damping



$$
\begin{aligned}
w_{tt}(\xi,t) + Ew_{\xi\xi\xi\xi}(\xi,t) + C_d w_{\xi\xi\xi\xi t}(\xi,t) &= 0\\
w(0,t) = 0, \quad Ew_{\xi\xi}(1,t) + C_d w_{\xi\xi t}(1,t) &= u(t)\\
w_\xi(0,t) = 0, \quad Ew_{\xi\xi\xi}(1,t) + C_d w_{\xi\xi\xi t}(1,t) &= 0\\
y(t) &= w_\xi(1,t)
\end{aligned}
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E, C_d are positive constants, $\xi \in (0,1)$ and $t > 0$

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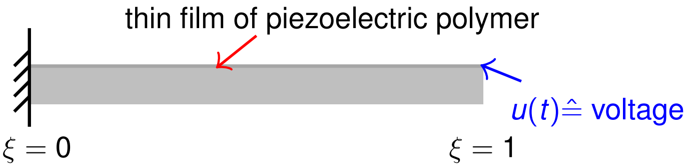

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with: $A_0 := E\frac{d^4}{d\xi^4}, \quad D = \frac{C_d}{E} A_0, \quad B_0 = \delta'(1),$
$z(t) \in H = L^2(0,1)$.

$$
z(t)(\xi) = w(\xi,t)
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Euler-Bernoulli beam equation: Kelvin-Voigt damping

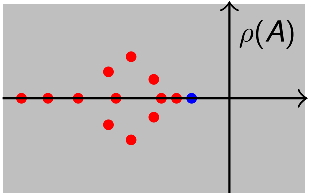
$$A_0 := E\frac{d^4}{d\xi^4}, \quad D = \frac{C_d}{E}A_0, \quad B_0 = \delta'(1),$$
$$D(A_0) = \{w \in H^4(0,1) \mid w(0)=w'(0)=w''(1)=w'''(1)=0\}.$$

$$\langle Dz, z\rangle = \frac{C_d}{E}\langle A_0 z, z\rangle = \frac{C_d}{E}||z||^2_{H_{1/2}}$$ "D is very large"

A_0 has compact resovent

$\sigma_{\text{ess}}(A) = \{-E/C_d\}$

$\sigma(A)\backslash\mathbb{R}$ consists of at most finitely many isolated normal eigenvalues



Transfer function of the second-order system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

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The transfer function of the system (A, B, C) is given by

$$G(s) = B_0^* \underbrace{\left(s^2 I + sD + A_0\right)^{-1}}_{\in \mathcal{L}(H_{-1/2}, H_{1/2})} B_0, \qquad s \in \rho(A).$$

$$G \in \mathcal{H}^\infty(\mathbb{C}_0; \mathbb{C}^{m\times m})$$

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When is a system minimum-phase?

Transfer function of the second order system

$$G(s) = B_0^*(s^2 I + sD + A_0)^{-1} B_0, \qquad s \in \mathbb{C}_0$$

Definition

System (A, B, C) is minimum-phase

$$\Longleftrightarrow$$

$G \in \mathcal{H}^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ and

$$\overline{\{G(\cdot)f(\cdot) \mid f \in \mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)\}}^{\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)} = \mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$$

$\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$ is the Hardy space on $\mathbb{C}_0$

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Why is miminum-phase important?

Infinite-dimensional systems

The minimum-phase property is important for

- PI-controller and high-gain control
  Logemann and Owens 1987, Logemann and Zwart 1992, Nikitin and Nikitina 1999, Kobayashi 2001, 2001, 2002,...
- Sensitivity minimization

It is important to know which systems are minimum-phase

Checkable condition for minimum-phase

Let $G \in \mathcal{H}^\infty(\mathbb{C}_0; \mathbb{C}^{m \times m})$.

Sufficient condition

Assume

- $s^n G(s) \not\to 0$ as $s \to \infty$ for some n
- G can be extended analytically over the imaginary axis.

Then

$$G \text{ minimum-phase} \iff \det G \text{ has no zeros in } \mathbb{C}_0$$

Checkable condition for minimum-phase

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Sufficient condition

Assume

- $s^n G(s) \not\to 0$ as $s \to \infty$ for some n
- G can be extended analytically over the imaginary axis.

Then

$$G \text{ minimum-phase} \iff \det G \text{ has no zeros in } \mathbb{C}_0$$

Minimum-phase behaviour: Second order system

Transfer function of the second order system

$$G(s) = B_0^*(s^2 I + sD + A_0)^{-1} B_0, \qquad s \in \mathbb{C}_0$$

Main result 1

If

- B_0 is injective,
- $\langle Dz, z\rangle \geq \beta \|z\|^2$ for any $z \in H_{\frac{1}{2}}$ for some $\beta > 0$.

then

G is minimum-phase and $\det G$ has no zeros on $i\mathbb{R}$.

Transfer function of the Euler-Bernoulli beam is minimum-phase.

Minimum-phase behaviour: Second order system

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Transfer function of the second order system

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Main result 2

If

- B_0 is injective,
- A_0^{-1} is compact,
- $\langle Dz, z\rangle > 0$ for any eigenvector z of A_0.

then

G is minimum-phase and $\det G$ has no zeros on $i\mathbb{R}$.

Thank you